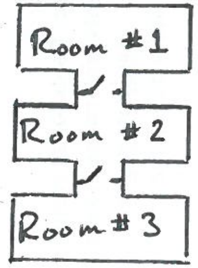
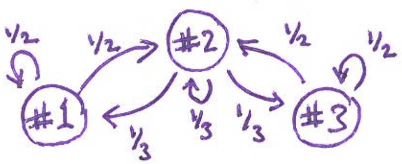
EX:

Three rooms are arranged in a line. Every minute people will, with equal probability, either move to an adjacent room or stay in their current room.



If 140 people begin all in room #1, calculate the expected number of people in each room after n minutes.

Transition Graph:



Transition Matrix:

$$T = \begin{array}{c} \begin{array}{ccc} \#1 & \#2 & \#3 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{2} \end{bmatrix} \end{array} \begin{array}{l} \to \#1 \\ \to \#2 \\ \to \#3 \end{array}$$

Initial State: $v_0 = \begin{bmatrix} 140 \\ 0 \\ 0 \end{bmatrix}$

Goal: n^{th} state vector $v_n = T\,v_{n-1} = T^2 v_{n-2} = \cdots$
$= T^n v_0$

Plan: Use eigenvalues & eigenvectors of T

$\to$ If $v_0 = \underbrace{v_1 + v_2 + v_3 + \cdots}_{\text{eigenvect.}}$

then $T^n v_0 = T^n v_1 + T^n v_2 + T^n v_3 + \cdots$
$= \lambda_1^n v_1 + \lambda_2^n v_2 + \lambda_3^n v_3 + \cdots$

First we must compute eigenval. & eigenvect. of

$$T = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{2} \end{bmatrix}$$

Bad news: 3×3 matrix so char. eqn. is degree 3... Difficult to factor ??

Good news: Markov matrix so $\lambda=1$ is an eigenvalue... Char. eqn. is $\underline{(\lambda-1)}$(degree 2 stuff) $=0$!

Important Simplification:

To remove fractions, we can multiply T by 6.

$T v = \lambda v$
$\Updownarrow$
$6T v = 6\lambda v$

This multiplies eigenvals by 6 and does not change eigenvect.

(Know: $6\lambda = 6\cdot 1 = 6$ is an eigenvalue.)

$$T = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{2} \end{bmatrix} \rightsquigarrow 6T = \begin{bmatrix} 3 & 2 & 0 \\ 3 & 2 & 3 \\ 0 & 2 & 3 \end{bmatrix}$$

Eigenvalues of $6T$: ← $= 6 \cdot$ Eigenvalues of T

Expand down column 1.

$$0 = \det \begin{bmatrix} \boxed{\begin{matrix} 3-\lambda \\ 3 \\ 0 \end{matrix}} & \begin{matrix} 2 \\ 2-\lambda \\ 2 \end{matrix} & \begin{matrix} 0 \\ 3 \\ 3-\lambda \end{matrix} \end{bmatrix}$$

$$0 = (3-\lambda) \det \begin{bmatrix} 2-\lambda & 3 \\ 2 & 3-\lambda \end{bmatrix} - 3 \det \begin{bmatrix} 2 & 0 \\ 2 & 3-\lambda \end{bmatrix} + 0$$

$$0 = \underline{(3-\lambda)}(\lambda^2 - 5\lambda) - 3(2\underline{(3-\lambda)})$$

$$0 = \underline{(3-\lambda)}\left((\lambda^2 - 5\lambda) - 3(2)\right)$$

$$0 = (3-\lambda)(\lambda-6)(\lambda+1) \Longrightarrow \boxed{\lambda = -1,\ 3,\ 6}$$

$$\left[\text{Eigenvalues of } T\text{: } \lambda/6 = -1/6,\ 3/6,\ 6/6 = -1/6,\ 1/2,\ 1 \right]$$

Eigenvectors of $6T$: ← $=$ Eigenvectors of T

$\lambda = -1$

$$\begin{bmatrix} 3-(-1) & 2 & 0 \\ 3 & 2-(-1) & 3 \\ 0 & 2 & 3-(-1) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 4x + 2y = 0 \\ 3x + 3y + 3z = 0 \\ -2y + 4z = 0 \end{cases}$$

let $x = 1$: ↳ $y = -2$ ↳ $z = 1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$\lambda = 3$

$$\begin{bmatrix} 3-3 & 2 & 0 \\ 3 & 2-3 & 3 \\ 0 & 2 & 3-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 2y = 0 \\ 3x - y + 3z = 0 \\ 2y = 0 \end{cases}$$

let $x = 1$: ↳ $y = 0$ ↳ $z = -1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = b \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\lambda = 6$ ← Note: This corresponds to $\lambda = 1$ of T. This eigenvector is the stable state.

$$\begin{bmatrix} 3-6 & 2 & 0 \\ 3 & 2-6 & 3 \\ 0 & 2 & 3-6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -3x + 2y = 0 \\ 3x - 4y + 3z = 0 \\ 2y - 3z = 0 \end{cases}$$

let $x = 2$: ↳ $y = 3$ ↳ $z = 2$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c \cdot \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

So far: T has eigenvectors & eigenvalues

- $v_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ with $\lambda_1 = -1/6$
- $v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ with $\lambda_2 = 1/2$
- $v_3 = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$ with $\lambda_3 = 1$

Now we must write the initial state vector as a sum of eigenvectors of T.

→ It would be nice if

$$\begin{bmatrix}140\\0\\0\end{bmatrix} = \begin{bmatrix}1\\-2\\1\end{bmatrix} + \begin{bmatrix}1\\0\\-1\end{bmatrix} + \begin{bmatrix}2\\3\\2\end{bmatrix}$$

... but it doesn't... we need different multiples

$$\begin{bmatrix}140\\0\\0\end{bmatrix} = a\begin{bmatrix}1\\-2\\1\end{bmatrix} + b\begin{bmatrix}1\\0\\-1\end{bmatrix} + c\begin{bmatrix}2\\3\\2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}140\\0\\0\end{bmatrix} = \begin{bmatrix}1 & 1 & 2\\-2 & 0 & 3\\1 & -1 & 2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}$$

(We could use LU decomposition to solve... but this time let's just convert to system.)

$$\begin{cases} 140 = a + b + 2c & (1)\\ 0 = -2a + 3c & (2)\\ 0 = a - b + 2c & (3)\end{cases}$$

Adding (1) and (3) gives

$140 = 2a + 4c$

Adding this to (2) gives

$140 = 7c$

$\hookrightarrow c = 20 \Rightarrow \begin{matrix} a = 30\\ b = 70\end{matrix}$

$$\begin{bmatrix}140\\0\\0\end{bmatrix} = \begin{bmatrix}30\\-60\\30\end{bmatrix} + \begin{bmatrix}70\\0\\-70\end{bmatrix} + \begin{bmatrix}40\\60\\40\end{bmatrix}$$ ← Sum of eigenvect!!

$$v_n = T^n\begin{bmatrix}140\\0\\0\end{bmatrix}$$

$$= T^n\left(\begin{bmatrix}30\\-60\\30\end{bmatrix} + \begin{bmatrix}70\\0\\-70\end{bmatrix} + \begin{bmatrix}40\\60\\40\end{bmatrix}\right)$$

$$= T^n\begin{bmatrix}30\\-60\\30\end{bmatrix} + T^n\begin{bmatrix}70\\0\\-70\end{bmatrix} + T^n\begin{bmatrix}40\\60\\40\end{bmatrix}$$

$$\boxed{= \left(-\tfrac{1}{6}\right)^n\underbrace{\begin{bmatrix}30\\-60\\30\end{bmatrix}}_{(-1/6)\text{ eigenvect.}} + \left(\tfrac{1}{2}\right)^n\underbrace{\begin{bmatrix}70\\0\\-70\end{bmatrix}}_{(1/2)\text{ eigenvect.}} + 1^n\underbrace{\begin{bmatrix}40\\60\\40\end{bmatrix}}_{(1)\text{ eigenvect.}}}$$

Check: When $n=0$ $\quad v_0 = \begin{bmatrix}30\\-60\\30\end{bmatrix} + \begin{bmatrix}70\\0\\-70\end{bmatrix} + \begin{bmatrix}40\\60\\40\end{bmatrix} \stackrel{\checkmark}{=} \begin{bmatrix}140\\0\\0\end{bmatrix}$

As $n \to \infty$ $\quad v_\infty = 0 + 0 + \begin{bmatrix}40\\60\\40\end{bmatrix} \stackrel{\checkmark}{=} v_s$

Eigenvectors with $\lambda \neq 1$ add to 0

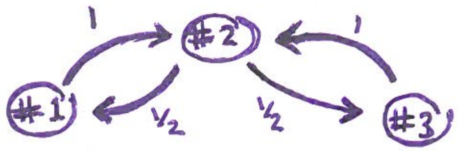
$\begin{bmatrix}1\\-2\\1\end{bmatrix}$ $\hookrightarrow 1-2+1 \stackrel{\checkmark}{=} 0$

$\begin{bmatrix}1\\0\\-1\end{bmatrix}$ $\hookrightarrow 1+0-1 \stackrel{\checkmark}{=} 0$

Let's do another example, more quickly this time.

Ex: Same setup as before (3 adjacent rooms) but this time do not allow people to stay in their room. 100 people start in room #1.

Transition Graph:



Transition Matrix:

$$T = \begin{array}{c} \begin{array}{ccc} \#1 & \#2 & \#3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} 0 & 1/2 & 0 \\ 1 & 0 & 1 \\ 0 & 1/2 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \to \#1 \\ \to \#2 \\ \to \#3 \end{array}$$

Find eigenvalues & eigenvectors of $2T = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$

(**Note**: One eigenvalue should be $2\lambda = 2 \cdot 1 = 2$)

$$0 = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 2 & -\lambda & 2 \\ 0 & 1 & -\lambda \end{bmatrix}$$ Expand across row 1

$$0 = -\lambda \cdot \det \begin{bmatrix} -\lambda & 2 \\ 1 & -\lambda \end{bmatrix} - 1 \cdot \det \begin{bmatrix} 2 & 2 \\ 0 & -\lambda \end{bmatrix} + 0$$

$$0 = \underline{-\lambda}\left(\lambda^2 - 2\right) - 2\underline{(-\lambda)}$$

$$0 = \underline{(-\lambda)}\left((\lambda^2 - 2) - 2\right)$$

$$0 = -\lambda(\lambda+2)(\lambda-2) \qquad \lambda = 0, -2, 2$$

[Eigenvalues of T $\lambda/2 = 0, -1, 1$]

$$\lambda = 0 \quad \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightsquigarrow \begin{cases} y = 0 \\ 2x + 2z = 0 \\ y = 0 \end{cases}$$

let $x = 1$: $\hookrightarrow y = 0$ $\hookrightarrow z = -1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\lambda = -2 \quad \begin{bmatrix} 0-(-2) & 1 & 0 \\ 2 & 0-(-2) & 2 \\ 0 & 1 & 0-(-2) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{cases} 2x + y = 0 \\ 2x + 2y + 2z = 0 \\ y + 2z = 0 \end{cases}$$

let $x = 1$: $\hookrightarrow y = -2$ $\hookrightarrow z = 1$

$$b \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\lambda = 2 \quad \begin{bmatrix} 0-2 & 1 & 0 \\ 2 & 0-2 & 2 \\ 0 & 1 & 0-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{cases} -2x + y = 0 \\ 2x - 2y + 2z = 0 \\ y - 2z = 0 \end{cases}$$

let $x = 1$: $\hookrightarrow y = 2$ $\hookrightarrow z = 1$

$$c \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Write initial state as sum of eigenvectors

$$\begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\left.\begin{aligned} 100 &= a + b + c \\ 0 &= -2b + 2c \\ 0 &= -a + b + c \end{aligned}\right\} \text{ Add} \rightarrow \quad \begin{aligned} 100 &= 2b + 2c \\ 0 &= -2b + 2c \\ \hline 100 &= 4c \Rightarrow c = 25 \end{aligned}$$

$b = 25$

$a = 50$

$$\begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \\ -50 \end{bmatrix} + \begin{bmatrix} 25 \\ -50 \\ 25 \end{bmatrix} + \begin{bmatrix} 25 \\ 50 \\ 25 \end{bmatrix}$$

$$\Downarrow$$

$$T^n \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} = \cancel{0^n \begin{bmatrix} 50 \\ 0 \\ -50 \end{bmatrix}} \rightarrow 0 \text{ if } n \neq 0 \; + (-1)^n \begin{bmatrix} 25 \\ -50 \\ 25 \end{bmatrix} + (1)^n \begin{bmatrix} 25 \\ 50 \\ 25 \end{bmatrix}$$

$$\boxed{= \begin{cases} \begin{bmatrix} 50 \\ 0 \\ 50 \end{bmatrix} & \text{if } n \text{ is even } (n \neq 0) \\ \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix} & \text{if } n \text{ is odd} \end{cases}}$$

The solution to this problem is very unusual. Instead of approaching the stable state vector, the state vector is bouncing back and forth between two possibilities (everyone in middle, and people split at ends). This happens because of the unusual eigenvalue $\lambda = -1$.